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Coherent states in geometric quantization

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Abstract

In this paper we study overcomplete systems of coherent states associated to compact integral symplectic manifolds by geometric quantization. Our main goals are to give a systematic treatment of the construction of such systems and to collect some recent results. We begin by recalling the basic constructions of geometric quantization in both the Kähler and non-Kähler cases. We then study the reproducing kernels associated to the quantum Hilbert spaces and use them to define symplectic coherent states. The rest of the paper is dedicated to the properties of symplectic coherent states and the corresponding Berezin–Toeplitz quantization. Specifically, we study overcompleteness, symplectic analogues of the basic properties of Bargmann's weighted analytic function spaces, and the 'maximally classical' behavior of symplectic coherent states. We also find explicit formulas for symplectic coherent states on compact Riemann surfaces.

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1. Introduction

Coherent states are ubiquitous in the mathematical physics literature. Yet there seems to be a lack of general theory in the context of geometric quantization. This paper is an attempt to partially fill this gap.

We will define coherent states associated to an arbitrary integral compact symplectic manifold (M, ω) using the machinery of geometric quantization. The metaplectic correction will not play a role in this construction and will be omitted for simplicity. In the non-Kähler case there are at least two common methods of quantizing M: almost Kähler [6,24] and Spin^c [13,23,25,29] quantization. The definition of coherent states which we will describe is similar in both cases, although for technical reasons is somewhat simpler in the almost Kähler case. In contrast to the quantization of a Kähler manifold, in both Spin^c and almost Kähler quantization a quantum state does not necessarily have a nice holomorphic local form. As a consequence, it is difficult to control the global behavior of the quantum states and, as we will see, the condition that M is compact becomes essential. This is not to say that the properties of the coherent states are different in the non-compact case — it is simply not clear to the author how to proceed (see [24] for some recent progress in this direction).

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The definition of coherent states that we will make is semi-constructive. It will depend on a choice of basis for the quantum Hilbert space arising from geometric quantization. Since such a basis is not always available, finding an explicit form for the corresponding coherent states may be difficult. On the other hand, the abstract approach taken here demonstrates that many of the traditional properties of coherent states follow from general considerations. We will refer to the coherent states constructed here as symplectic coherent states in order to distinguish them from specific instances.

We should point out that symplectic coherent states are not, in general, of Perelomov-type [27]; i.e. they are not orbits of a fiducial vector under the action of a Lie group. In some specific cases symplectic coherent states and Perelomov coherent states coincide — for example in the quantizations of $\mathbb{C}$ and S^2 (see Section 6), where the methods of Perelomov yield a reproducing system in the quantum Hilbert space that arises from geometric quantization. In fact, when this happens, the two constructions must agree (Theorem 4).

The coherent state map we will study has appeared in a different (but in some ways equivalent) form in [7] where it is used to prove a symplectic analogue of Kodaira's embedding theorem. In [7], the coherent states lie in a different quantum Hilbert space and are associated to a circle bundle over M. In [24], Ma–Marinescu analyze the semiclassical properties of generalized Bergman kernels. This analysis leads to the notion of a peak section, which is related to symplectic coherent states (Section 3.2). In [28], Rawnsley globalized the constructions of Perelomov [27] to make a general definition of a coherent state on Kähler manifolds. When applied to the specific case of a principal $\mathbb{C}^\times$-bundle over M the coherent states constructed here and the Rawnsley type coherent states are related (see Section 3.3). The properties of Rawnsley-type coherent states, as well as the geometric interpretation of Berezin quantization which they provide, are studied in [9,10]. Symplectic coherent states are associated to M itself, and can be associated to a non-Kähler symplectic manifold.

Symplectic coherent states generalize many of the systems constructed in the literature. The most basic examples of coherent states are those associated to the complex plane — the simplest Kähler manifold. These states are known as Segal–Bargmann–Heisenberg–Weyl (or some permutation thereof) coherent states, or often more simply as canonical coherent states; see [12], for example, where they are developed using projective representations of the symplectic group on the quantum Hilbert spaces of Section 2.2. Canonical coherent states are briefly described in Section 6. Ref. [17] contains a survey on many traditional mathematical aspects of canonical coherent states, the introduction of which also includes a discussion of what, in general, should be called a coherent state.

Following [17] (and to some extent popular opinion) we will define a system of coherent states to be a set $\{|x\rangle \in \mathcal{H} \mid x \in M\}$ of quantum states in some quantum Hilbert space $\mathcal{H}$, parameterized by some set M, such that:

(1) the map $x \mapsto |x\rangle$ is smooth, and
(2) the system is overcomplete; i.e.

$$\int_M |x\rangle\langle x| \,\mathrm{d}\mu(x) = \mathbf{1}_{\mathcal{H}}.$$

Physicists usually call property (2) completeness. As we will see, the map $x \mapsto |x\rangle$ is actually antiholomorphic in a sense appropriate to non-Kähler manifolds. The parameterizing set M is generally, and for us will be, a classical phase space; i.e. an integral symplectic manifold.

We motivate our construction of coherent states by recalling some basic quantum mechanics. The following observations are well known ([17] and [20, Chapter 3]). Let us, for a moment, eschew definitions and rigor in order to see how to proceed. The position space wave function representing a state $|\psi\rangle$ is $\psi(x) = \langle x|\psi\rangle$, where $|x\rangle$ is a coherent state localized at x. The position space wave function of the coherent state $|x\rangle$ is then

$$K(x, y) := K_x(y) := \langle y|x\rangle.$$

We can use this to rewrite the equation $\langle x|\psi\rangle = \psi(x)$ as

$$\int \overline{K(x, y)}\psi(y)\,\mathrm{d}y = \psi(x). \tag{1}$$

A function K that satisfies (1) for some space of functions is called a reproducing kernel for that space. Reading the above discussion backwards, we see that a coherent state can be defined in terms of a reproducing kernel. We will use this approach and define symplectic coherent states in terms of reproducing kernels for the quantum Hilbert spaces arising from geometric quantization, sometimes called generalized Bergman kernels; properties (1) and (2) will then follow. This construction is well known for the Kähler quantization of $\mathbb{C}$ and yields the Bergman reproducing kernel, which in turn yields the canonical coherent states. The asymptotics of generalized Bergman kernels are studied in [6, 7,11,23,24] and will be important when we consider the semiclassical limit.

Example 1. A reproducing kernel for the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ is the Dirac distribution $\delta(x-y)$.

Symplectic coherent states associated to the Poincaré disc, and hence via Riemann uniformization to compact Riemann surfaces of genus $g \geq 2$, were used in [18,19] to study the semiclassical limit of the deformation quantizations of these surfaces. We will give explicit formulas for these coherent states in Section 6.

The rest of the paper is organized as follows: in Section 2 we review the geometric quantization of a Kähler manifold (M, ω) and two generalizations to the non-Kähler case known as the almost Kähler and Spin^c quantizations of (M, ω). In Section 3 we define the reproducing kernel for the quantum Hilbert space associated to M by geometric quantization and use it to define symplectic coherent states. We also describe symplectic analogues of some analytic function space results of [2], and the relationship between Rawnsley-type and symplectic coherent states. In Section 4 we discuss the overcompleteness relation and the coherent state quantization induced by the symplectic coherent states. In Section 5 we show that symplectic coherent states are the most classical quantum states and consider the semiclassical limit. Finally, in Section 6 we apply the constructions of Section 3 to compact Riemann surfaces.

2. Background and notation

2.1. Prequantization

Throughout we assume that (M, ω) is an integral compact symplectic manifold; i.e. $\left[\frac{\omega}{2\pi}\right]$ is in the image of the map $H^2(M;\mathbb{Z}) \to H^2_{\mathrm{DR}}(M)$. The basic object of geometric quantization is an Hermitian line bundle $\pi : \ell \to M$ with compatible connection ∇ with curvature $-i\omega$, known as the prequantum line bundle. The existence of ℓ is guaranteed by the integrality of ω; in fact the Chern character of $\ell^{\otimes k}$ is $ch(\ell^{\otimes k}) = \exp(k\left[\frac{\omega}{2\pi}\right])$. For detailed accounts of geometric quantization see [14,31].

We denote by $h : \overline{\ell}_x \otimes \ell_x \to \mathbb{C}$ the Hermitian structure on ℓ. We follow the physics convention that the first term is conjugate linear. All tensor products will be taken over $\mathbb{C}$. The norm of $q \in L_x$ is $|q|^2 = h(q, q)$. h induces an Hermitian structure on $\Gamma(\ell)$: for $s_1, s_2 \in \Gamma(\ell)$

$$\langle s_1|s_2\rangle = \int_M h(s_1(x), s_2(x))\epsilon_\omega(x),$$

where

$$\epsilon_\omega = \left(\frac{1}{2\pi}\right)^n \frac{\omega^{\wedge n}}{n!}$$

is the Liouville volume form on M. We have included recurring factors of 2π in the Liouville form. This will simplify some formulas later on, but will also have the effect that our formulas differ slightly from some of those in the literature. The norm of $s \in \Gamma(\ell)$ is $\|s\|^2 = \langle s|s\rangle$.

We will be occasionally interested in the semiclassical $k = 1/\hbar \to \infty$ limit of $\ell^{\otimes k}$. The structures $(h, \langle\cdot|\cdot\rangle, \nabla)$ on ℓ induce corresponding structures on $\ell^{\otimes k}$ which we will denote by the same symbols. The curvature of the connection on $\ell^{\otimes k}$ is $-ik\omega$.

The program of geometric quantization associates to (M, ω) a Hilbert space $\mathcal{H}$ and a map $Q : C^\infty(M) \to \mathrm{Op}(\mathcal{H})$. To begin, we define the prequantum Hilbert space $\mathcal{H}^0_k$ to be the L^2 completion, with respect to the Liouville measure, of the set of square integrable sections of $\ell^{\otimes k}$:

$$\mathcal{H}^0_k = \overline{\{s \in \Gamma(\ell^{\otimes k}) \mid \|s\| < \infty\}}.$$

The Kostant–Souriau quantization of the Poisson–Lie algebra $C^\infty(M)$ is the map

$$f \in C^\infty(M) \mapsto Q^{(k)}_{\mathrm{KS}}(f) = -\frac{i}{k}\nabla_{X_f} + f \in \mathrm{Op}(\mathcal{H}^0_k)$$

where X_f is the Hamiltonian vector field defined by

$$X_f \lrcorner \omega = \mathrm{d}f.$$

In Section 4 we will recall an alternate quantization of $C^\infty(M)$.

As is well known, $\mathcal{H}_k^0$ is too large for the purposes of quantization [31, Chapter 9]. If (M, ω) is Kähler there is a standard method of choosing a subspace $\mathcal{H}_k \subset \mathcal{H}_k^0$. In the non-Kähler case, there are (at least) two reasonable methods: almost Kähler quantization [6,24] and Spin^c quantization [13,23,25,29]. We will review these three constructions in the next three sections.

2.2. Kähler quantization

If (M, ω, J) is a Kähler manifold with complex structure J, there is a natural method of reducing the prequantum Hilbert space $\mathcal{H}_k^0$. The complexified tangent bundle of M decomposes into $\pm i$ eigenspaces of J:

$$TM_{\mathbb{C}} = TM_J^{(1,0)} \oplus TM_J^{(0,1)}. \tag{2}$$

A section $s \in \Gamma(\ell^{\otimes k})$ is said to be polarized if it is tangent to the Kähler polarization $TM_J^{(1,0)}$; i.e. if $\nabla_X s = 0$ for each $X \in \Gamma(TM^{(0,1)})$.

The quantum Hilbert space is defined to be the L^2 closure of the set of polarized sections of $\ell^{\otimes k}$:

$$\mathcal{H}_k = \overline{\{s \in \Gamma(\ell^{\otimes k}) \mid \|s\| < \infty, s \text{ polarized}\}}.$$

This quantum Hilbert space can be described in terms of a Dirac-type operator [4, Chapter 3]. The decomposition (2) induces a decomposition

$$\Lambda^*(T^*M) = \bigoplus_{p,q=0}^{n} \Lambda^{p,q}(T^*M) = \bigoplus_{p,q=0}^{n} \Lambda^p(T^*M_J^{(1,0)}) \otimes \Lambda^q(T^*M_J^{(0,1)}). \tag{3}$$

Let $\overline{\partial}_k : \Omega^{p,q}(M, \ell^{\otimes k}) \to \Omega^{p,q+1}(M, \ell^{\otimes k})$ denote the Dolbeault operator twisted by $\ell^{\otimes k}$. Hodge's theorem says that $\ker(\overline{\partial}_k + \overline{\partial}_k^*)^2$ is isomorphic to the sheaf cohomology space $H^*(M, \mathcal{O}(\ell^{\otimes k}))$. Kodaira's vanishing theorem then tells us that for k sufficiently large, $H^q(M, \mathcal{O}(\ell^{\otimes k})) = 0$ for $q > 0$, and hence that $\mathcal{H}_k = \ker \overline{\partial}_k \mid_{\Gamma(\ell^{\otimes k})}$. The dimension of $\mathcal{H}_k$ can be computed with the Riemann–Roch–Hirzebruch theorem:

$$d_k := \dim \mathcal{H}_k = RR(M, \ell^{\otimes k}) = \int_M ch(\ell^{\otimes k}) T\mathrm{d}(TM_J^{(1,0)}). \tag{4}$$

In particular, since we assume M is compact, $d_k < \infty$.

2.3. Almost Kähler quantization

We suppose now that (M, ω) is an integral compact symplectic manifold, not necessarily Kähler. Every such manifold admits an ω-compatible almost complex structure J, and any two such choices are homotopic. The complexified tangent bundle again decomposes as in (2). If J is not integrable (i.e. M is not a complex manifold), then there may be no sections $s \in \Gamma(\ell^{\otimes k})$ which are tangent to $TM_J^{(1,0)}$. In this case, more work is required to define a quantum Hilbert space.

In this section, we describe one such method, known as almost Kähler quantization, which was introduced in [6] based on results in [15] and further studied in [11,23,24]. The idea is to replace $(\overline{\partial}_k + \overline{\partial}_k^*)^2$, which does not exist if J is not integrable, with the rescaled Laplacian $\Delta_k := \Delta - nk$, where Δ is the Laplacian for the metric $g = \omega(\cdot, J\cdot)$. In the Kähler case, these two quantities are equated by the Bochner–Kodaira formula: $\Delta_k = 2(\overline{\partial}_k + \overline{\partial}_k^*)^2$. The main result of [15] is (in a slightly sharpened form due to [6,24]):

Theorem 2. *Given an integral symplectic manifold (M, ω) with ω-compatible almost complex structure, there exists a constant C and a positive constant a such that for k sufficiently large,*

(1) *the first* d_k *eigenvalues of* Δ_k *(in nondecreasing order) lie in the interval* $(-a, a)$*, and*
(2) *the remaining eigenvalues lie to the right of* $nk + C$*,*

where d_k *is the Riemann–Roch number as in* (4)*.*

Motivated by this, the quantum Hilbert space is defined to be

$$\mathcal{H}_k := \text{span}_{\mathbb{C}}\{\theta_1^{(k)}, \dots, \theta_{d_k}^{(k)}\}$$

where $\theta_j^{(k)}$ is an eigenfunction of Δ_k with eigenvalue λ_j and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots$. Note that just as in the Kähler case, the dimension is given by the Riemann–Roch number d_k. This quantization has excellent semiclassical ($\hbar = 1/k \to 0$) properties [5,6,23,24]. It also has the advantage that the quantum Hilbert space consists of sections of $\ell^{\otimes k}$. Spin^c quantization, described in the next section, does not have either of these properties, although the Spin^c quantum states are true zero-modes of a Dirac-type operator.

In analogy with the Kähler case, we will call the elements of $\mathcal{H}_k$ polarized sections. Since M is compact and the polarized sections are smooth and the quantum Hilbert space is finite dimensional, $\mathcal{H}_k$ is a closed subspace of $L^2(M, \ell^{\otimes k})$.

2.4. Spin^c *quantization*

The idea of Spin^c quantization is to find a suitable generalization of $\overline{\partial}_k$ for the non-Kähler case. We will briefly review the relevant details here. See [22, App D] for a more complete account of the Spin^c bundle, and [13,23,25,29] for Spin^c quantization.

Let J be an ω-compatible almost complex structure on (M, ω) so that $g = \omega(\cdot, J\cdot)$ is a Riemannian metric on M. The Spin^c bundle associated to the data (M, ω, J) is defined as $S(M) := \Lambda^{0,*}(T^*M)$ according to the decomposition (3). There is a Dirac-type operator $\overline{\not{\partial}}_k : \Omega^{0,*}(M, \ell^{\otimes k}) \to \Omega^{0,*}(M, \ell^{\otimes k})$ that decomposes into $(\overline{\not{\partial}}_k)_+ : \Omega^{0,\text{even}}(M, \ell^{\otimes k}) \to \Omega^{0,\text{odd}}(M, \ell^{\otimes k})$ and $(\overline{\not{\partial}}_k)_- : \Omega^{0,\text{odd}}(M, \ell^{\otimes k}) \to \Omega^{0,\text{even}}(M, \ell^{\otimes k})$.

The quantum Hilbert space associated to the data (M, ω, J) is the virtual vector space

$$\mathcal{H}_k := \ker(\overline{\not{\partial}}_k)_+ \ominus \ker(\overline{\not{\partial}}_k)_-.$$

The dimension is again given by the Riemann–Roch number d_k (4). There is a Spin^c analogue of the Kodaira vanishing theorem [6,23] which insures that $\mathcal{H}_k$ is an honest vector space for k sufficiently large.

The metric g on M and the Hermitian structure h on $\ell^{\otimes k}$ combine to give an Hermitian structure, also denoted by h, on $S(M)$. Although the zero-modes of $\overline{\not{\partial}}_k$ are not sections of $\ell^{\otimes k}$ since they have higher degree components, their norms are asymptotically concentrated on the zero degree part; i.e. there exists a constant $C > 0$ such that for k sufficiently large $\|s_+\| \le Ck^{-1/2}\|s_0\|$, for each $s \in \mathcal{H}_k$ and where $s = s_0 + s_+$ denotes the decomposition of s into zero and higher degree components [6,23].

Just as in the Kähler and almost Kähler cases, we will refer to the elements of $\mathcal{H}_k$ as polarized sections. Also, since M is compact and polarized sections are again smooth, and the space of them is finite dimensional, $\mathcal{H}_k$ is a closed subspace of $L^2(M, S(M))$.

We will assume throughout that k is chosen sufficiently large to ensure the validity of the relevant vanishing/existence theorem.

3. Coherent states

In this section we will construct coherent states associated to an integral symplectic manifold (M, ω).

3.1. Reproducing kernels

For the Kähler, almost Kähler and Spin^c quantizations of a compact symplectic manifold (M, ω) the quantum Hilbert space is finite dimensional with dimension given by the Riemann–Roch formula (4). We will see below that since $d_k < \infty$ there exists a reproducing kernel for the quantum Hilbert space. In the Kähler case, the existence of a reproducing kernel may also be established by trivializing $\ell^{\otimes k}$ — polarized sections are locally holomorphic and standard methods from complex analysis can be used. In the non-Kähler cases no such nice local form is known to the author, and the compactness assumption and resulting finite dimensionality become essential.

For two vector bundles $\pi_j : E_j \to M_j,\ j = 1, 2$ we define $E_1 \boxtimes E_2 := \pi_1^* E_1 \otimes \pi_2^* E_2 \to M_1 \times M_2$. If a vector bundle E has an Hermitian structure h, we will identify $\overline{E} \simeq E^*$. For $u, v \in E_x$ we define $\overline{u} \cdot v := h(u, v)$. Similarly, we identify $\overline{u} \otimes u = h(u, u)$. Moreover, if L is a line bundle, we will identify $\overline{v} \otimes w = h(v, w)$ for $\overline{v} \in \overline{L}_x,\ w \in L_x$ so that $\overline{L} \otimes L \simeq \mathbb{C}$. Combining these definitions, we see that $u \otimes \overline{v} \cdot w = h(v, w)u$. These conventions agree since $u \otimes \overline{v} \otimes w = h(v, w)u = h(v, u)w = u \otimes \overline{v} \cdot w$. Finally, we define $\overline{\overline{u} \otimes v} := u \otimes \overline{v}$.

Many of the results in this section hold for all three methods of quantization. To unify notation and avoid repetition we define

$$L^k := \begin{cases} \ell^{\otimes k} & \text{for Kahler and almost Kahler quantization} \\ \Lambda^{0,*}(T^*M) \otimes \ell^{\otimes k} & \text{for Spin}^c \text{ quantization.} \end{cases}$$

Let $\{\theta_j^{(k)}\}_{j=1}^{d_k}$ be a unitary basis for $\mathcal{H}_k$.

Definition 3. The reproducing kernel $K^{(k)} \in \Gamma(\overline{L^k} \boxtimes L^k)$ is the section

$$K^{(k)}(x, y) := \sum_{j=1}^{d_k} \overline{\theta_j^{(k)}(x)} \otimes \theta_j^{(k)}(y).$$

$K^{(k)}$ is also known as a generalized Bergman kernel. Note that $K^{(k)}$ does not depend on the choice of unitary basis.

In the Kähler and Spin^c quantization schemes, the quantum Hilbert space is the kernel of a Dirac-type operator, and the reproducing kernel is the large t limit of the associated heat kernel (see Section 4.2). Although it is not a function on $M \times M$, the reproducing kernel $K^{(k)}$ has many of the same properties enjoyed by reproducing kernels for analytic function spaces.

Theorem 4. *$K^{(k)}$ is the unique polarized section of $\overline{L^k} \boxtimes L^k$ such that*

$$\int_M \overline{K^{(k)}(x, y)} \cdot s(y)\, \epsilon_\omega(y) = s(x) \quad \forall s \in \mathcal{H}_k.$$

Proof. Suppose there are two reproducing kernels. Their difference evaluated against an arbitrary section $s \in \mathcal{H}_k$ must be zero. Hence this difference must be in $\overline{\mathcal{H}_k} \otimes (\mathcal{H}_k)^\perp$. But since both kernels are polarized, the difference is in $\overline{\mathcal{H}_k} \otimes \mathcal{H}_k$ and is therefore zero. □

The restriction of $K^{(k)}$ to the diagonal is a smooth function.

Definition 5. The coherent density is the smooth function $\varepsilon^{(k)} \in C^\infty(M)$ defined by

$$\varepsilon^{(k)}(x) := K^{(k)}(x, x) = \sum_{j=1}^{d_k} |\theta_j^{(k)}(x)|^2.$$

Since M is compact and $\varepsilon^{(k)}$ is smooth and nonnegative, we may define a measure on M by $\mu^{(k)} = \varepsilon^{(k)} \epsilon_\omega$ which we will call the coherent measure.

Since $\mathcal{H}_k \subseteq L^2(M, L^k)$ is a closed subspace, there is a projection $\Pi_k : L^2(M, L^k) \to \mathcal{H}_k$. To find the Schwartz kernel of this projection, we need the following observation (which follows from the facts that $L^2(\mathbb{R}^{2n}, \mathbb{C})$ is separable and that since M is compact it has a finite cover by open sets which are diffeomorphic to subsets of $\mathbb{R}^{2n}$).

Theorem 6. *$L^2(M, L^k)$ is a separable Hilbert space.*

Theorem 7. *The Schwartz kernel of Π_k is the reproducing kernel $K^{(k)}$.*

Proof. We need to show that

$$(\Pi_k s)(x) = \int_M \overline{K^{(k)}(x,y)} \cdot s(y)\epsilon_\omega(y) \qquad \forall s \in L^2(M, L^k).$$

Since Π_k is a projection, it is uniquely characterized by im $\Pi_k = \mathcal{H}_k$ and $\Pi_k^2 = \Pi_k = \mathbf{1}_{\mathcal{H}_k}$.

Let $\{\theta_j^{(k)}\}_{j=1}^\infty$ be a unitary basis for $L^2(M, L^k)$ such that $\text{span}_\mathbb{C}\{\theta_j^{(k)}\}_{j=1}^{d_k} = \mathcal{H}_k$. Then for each $s \in L^2(M, L^k)$ there exists $\{s^j \in \mathbb{C}\}_{j=1}^\infty$ such that $\|s - \sum_{j=1}^N s^j \theta_j^{(k)}\|^2 \to 0$ as $N \to \infty$. We then have

$$\int_M \overline{K^{(k)}(x,y)} \cdot s(y)\epsilon_\omega(y) = \sum_{j=1}^{d_k} \theta_j^{(k)}(x) \int_M \sum_{l=1}^\infty s^l h(\theta_j^{(k)}(y), \theta_l^{(k)}(y))\epsilon_\omega(y). \tag{5}$$

Using Hölder's inequality we see that

$$\sum_{l=1}^\infty |s^l|^2 \int_M |h(\theta_j^{(k)}(y), \theta_l^{(k)}(y))|\epsilon_\omega(y) \le \sum_{l=1}^\infty |s^l|^2 \|\theta_j^{(k)}(y)\|^2 \|\theta_l^{(k)}(y)\|^2 = \|s\|^2.$$

Hence, the integrand is absolutely integrable and we may interchange the integral and sum in (5) to obtain

$$\int_M \overline{K^{(k)}(x,y)} \cdot s(y)\epsilon_\omega(y) = \sum_{j=1}^{d_k} \theta_j^{(k)}(x) \in \mathcal{H}_k$$

as desired.

Moreover, for $s = \sum_{j=1}^{d_k} s^j \theta_j^{(k)}(x) \in \mathcal{H}_k$, we easily obtain $\Pi_k^2 s = \Pi_k s$ in terms of $K^{(k)}$ since all of the relevant sums are finite. We conclude that $K^{(k)}$ is the Schwartz kernel of Π_k as desired. □

3.2. Coherent states

We now define the coherent states associated to an integral compact symplectic manifold (M, ω).

Definition 8. The coherent state localized at $x \in M$ is

$$\Phi_x^{(k)} := K^{(k)}(x, \cdot) = \sum_{j=1}^{d_k} \overline{\theta_j^{(k)}(x)} \otimes \theta_j^{(k)}.$$

In order to distinguish these coherent states from others, we will sometimes refer to them as symplectic coherent states. Observe that $\Phi^{(k)}$ depends smoothly and antiholomorphically on x in the generalized sense: a section is holomorphic on a symplectic manifold if it is polarized.

Since $\Phi_x^{(k)} \in \overline{L_x^k} \otimes \mathcal{H}_k$, it is necessary to investigate how $\Phi_x^{(k)}$ should be interpreted as a quantum state. Consider the case of almost Kähler quantization. If we trivialize $\ell_x^{\otimes k}$ with a unit, $\Phi_x^{(k)}$ becomes a well-defined state in $\mathcal{H}_k$ via the identification $1 \otimes \theta_j^{(k)} \simeq \theta_j^{(k)}$. The different unit trivializations of $\ell_x^{\otimes k}$ are parameterized by $U(1)$ which means that $\Phi_x^{(k)}$ is a well-defined quantum state up to a phase — the usual situation in quantum mechanics. On the other hand, quantum states are most properly regarded as rays in the projective Hilbert space $\mathbb{P}\mathcal{H}_k$. It follows from the above discussion that the map $x \in M \mapsto \mathbb{C} \cdot \Phi_x^{(k)} \in \mathbb{P}\mathcal{H}_k$ is well-defined and smooth. If it is not possible to find a global unit section of $\ell^{\otimes k}$ then there is no smooth lift of $\Phi^{(k)}$ to $\mathcal{H}_k$.

In [24], this map is shown to be asymptotically symplectic (as $k \to \infty$), asymptotically isometric with respect to the metric $g = \omega \circ J$, and, for k sufficiently large, an embedding (see [7] for similar results).

We now return our attention to the general case of almost Kähler or Spin^c quantization. It is sometimes convenient to work with normalized states. In terms of coherent states, Definition 5 reads

Theorem 9. $\|\Phi_x^{(k)}\|^2 = \varepsilon^{(k)}(x)$.

This is the reason $\varepsilon^{(k)}$ is called the coherent density.

Let $M_k := \{x \in M \mid \varepsilon^{(k)}(x) \neq 0\}$. By Corollary 12, M_k is the complement of the base locus of $\mathcal{H}_k$. For $x \in M_k$ we will denote the normalized coherent state localized at x by

$$\widetilde{\Phi}_x^{(k)} := |x^{(k)}\rangle := \frac{\Phi_x^{(k)}}{\sqrt{\varepsilon^{(k)}(x)}}.$$

For $x \notin M_k$ we define $\widetilde{\Phi}_x^{(k)} = |x^{(k)}\rangle := 0$. We can now state the reproducing property concisely:

$$\langle \Phi_x^{(k)}|s\rangle = s(x)$$
$$\sqrt{\varepsilon^{(k)}(x)}\langle x^{(k)}|s\rangle = s(x) \quad \forall x \in M, \forall s \in \mathcal{H}_k.$$

For $x \notin M_k$ the above is justified by Corollary 12.

If $x \in M_k$ then $\mathbb{C} \cdot \Phi_x^{(k)} = \mathbb{C} \cdot |x^{(k)}\rangle \in \mathbb{P}\mathcal{H}_k$ so that we may define the projection $\mathrm{proj}_{|x^{(k)}\rangle} : \mathcal{H}_k \to \mathcal{H}_k$ by

$$s \mapsto (\varepsilon^{(k)}(x))^{-1}|\Phi_x^{(k)}\rangle\langle \Phi_x^{(k)}|s\rangle = |x^{(k)}\rangle\langle x^{(k)}|s\rangle. \tag{6}$$

We will also write $\mathrm{proj}_{\mathbb{C}\cdot\Phi_x^{(k)}} = |x^{(k)}\rangle\langle x^{(k)}| = (\varepsilon^{(k)}(x))^{-1}|\Phi_x^{(k)}\rangle\langle \Phi_x^{(k)}|$ for this projection.

The following is a generalization of a result in [2] and is the basic reason why the quantum Hilbert space behaves in many ways like a weighted analytic function space, even on non-Kähler manifolds. We will further develop this analogy in Section 4.1.

Theorem 10. *For each polarized section $s \in \mathcal{H}_k$,*

$$|s(x)| \leq \|s\|\sqrt{\varepsilon^{(k)}(x)}.$$

Proof. Let $s \in \mathcal{H}_k$. We use the reproducing property of the coherent states to write, for $x \in M_k$,

$$|s(x)|^2 = h(s(x), s(x)) = h(\langle \Phi_x^{(k)}|s\rangle, \langle \Phi_x^{(k)}|s\rangle) = |\langle \Phi_x^{(k)}|s\rangle|^2.$$

The result then follows from the Cauchy–Schwartz inequality and Theorem 9.

If $x \notin M_k$ then $\varepsilon^{(k)}(x) = 0$ implies $\theta_j^{(k)}(x) =$ for all j. Since s is a linear combination of $\theta_j^{(k)}$, this implies $s(x) = 0$. □

This theorem has two useful corollaries. The proofs are simple and left to the reader.

Corollary 11. *The evaluation map* $\mathrm{ev}_x : s \in \mathcal{H}_k \mapsto s(x) \in L_x^k$ *is continuous.*

We can also prove Corollary 11 directly — it follows from the facts that $d_k < \infty$ and that the sets $\{|\theta_j^{(k)}(x)|\}_{j=1}^{d_k}$ and $\{\|\theta_j^{(k)}\|\}_{j=1}^{d_k}$ are bounded, which in turn follow from our assumption that M is compact.

In the Kähler case, the compactness assumption on M can be lifted. The existence of the reproducing kernel, as well as Theorem 10, can then be deduced from Jensen's formula [21, p. 324]: if f is holomorphic on the closed disc of radius R and $f(0) \neq 0$, and the zeroes of f in the open disc, ordered by increasing moduli and repeated according to multiplicity, are $z_1, \ldots, z_N$, then

$$|f(0)| \leq \frac{\|f\|_R}{R^N}|z_1 \cdots z_N|.$$

Unfortunately, the author is unaware of nice local forms of polarized sections in the Spin^c and almost Kähler quantizations of a non-Kähler manifold. This makes the assumption that M is compact, or more precisely that $d_k < \infty$, essential. Of course, it is also not clear whether the spectrum of the rescaled Laplacian has the requisite structure to define the almost Kähler quantization of M in the non-compact case.

On the other hand, most of the results of this paper hold for any choice of finite dimensional subspace of the prequantum Hilbert space $\mathcal{H}_k^0$. The primary advantages of almost Kähler and Spin^c quantization are that they provide canonical methods for choosing such a subspace and that they provide enough structure to ensure a meaningful semiclassical limit (cf. (15)).

The next corollary justifies our definition of the normalized coherent state $|x^{(k)}\rangle$ in the case that $\varepsilon^{(k)}(x) = 0$.

Corollary 12. *$\varepsilon^{(k)}(x) = 0$ if and only if $s(x) = 0$ for each polarized section s.*

In Sections 3.4, 5 and 6 we will be interested in the semiclassical limit of the symplectic coherent states. It will be useful to express $\Phi_x^{(k)}$ in terms of the peak sections of Ma–Marinescu [24], defined as follows. The Kodaira map $\Psi^{(k)} : M_k \to \mathbb{P}\mathcal{H}_k^*$, which sends $x \in M_k$ to the hyperplane $\{s \in \mathcal{H}_k \mid s(x) = 0\}$ of sections which vanish at x, is base point free for large enough k. Construct an unitary basis $\{\theta_1^{(k)}, \ldots, \theta_{d_k-1}^{(k)}, S_x^{(k)}\}$ such that $\theta_j^{(k)}(x) = 0$ for $1 \le j \le d_k - 1$. Then $S_x^{(k)}$, called a peak section, is a unit norm generator of the orthogonal complement of $\Psi^{(k)}(x)$. Observe that

$$\Phi_x^{(k)}(y) = K^{(k)}(x, y) = \overline{S_x^{(k)}(x)} \otimes S_x^{(k)}(y) \tag{7}$$

and also that $\varepsilon^{(k)}(x) = |S_x^{(k)}(x)|^2$. Moreover,

$$\int_M |S_x^{(k)}(y)|^2 \epsilon_\omega(y) = 1. \tag{8}$$

3.3. Rawnsley-type coherent states

The coherent states defined in [28] for compact Kähler manifolds are a generalization of Bargmann's principal vectors [2] to spaces of holomorphic sections. We will describe their relation to symplectic coherent states in this section. Due to Corollary 11, we are able to construct Rawnsley-type coherent states on any compact integral symplectic manifold. In this section, we will consider only the almost Kähler quantization of M so that the prequantum bundle is $\ell^{\otimes k}$.

By Corollary 11, for each $q \in \ell_x$ we get a continuous map $\delta_q : \mathcal{H}_k \to \mathbb{C}$ by composing the evaluation ev_x with the trivialization $s(x) = \delta_q(s) q^{\otimes k}$. By the Riesz representation theorem, there exists $e_q^{(k)} \in \mathcal{H}_k$ such that

$$\langle e_q^{(k)} | s\rangle q^{\otimes k} = \delta_q(s) q^{\otimes k} = s(\pi(q)) \qquad \forall s \in \mathcal{H}_k.$$

Observe that

$$e_{cq}^{(k)} = \bar{c}^{-k} e_q^{(k)}$$

for $0 \neq c \in \mathbb{C}$. We will refer to the section $e_q^{(k)}$ as a Rawnsley-type coherent state. We can express the reproducing kernel, and therefore the symplectic coherent states, in terms of $e_q^{(k)}$:

Theorem 13. *Let $x \in M$ and $q \in \ell_x$. Then*

$$K^{(k)}(x, y) = \bar{q}^{\otimes k} \otimes e_q^{(k)}(y).$$

Equivalently, $\Phi_x^{(k)} = \bar{q}^{\otimes k} \otimes e_q^{(k)}$.

Proof. In terms of the unitary basis $\{\theta_j^{(k)}\}_{j=1}^{d_k}$ for $\mathcal{H}_k$ we have

$$s(x) = \langle \Phi_x^{(k)} | s\rangle = \sum_{j=1}^{d_k} \langle \theta_j^{(k)} | s\rangle \theta_j^{(k)}(x) = \sum_{j=1}^{d_k} \langle \theta_j^{(k)} | s\rangle \tilde{\theta}_j^q(x) q^{\otimes k} = \langle e_q^{(k)} | s\rangle q^{\otimes k}$$

where $\tilde{\theta}_j^q$ is the trivialization of $\theta_j^{(k)}$ determined by a local section with value $q^{\otimes k}$ at x. Therefore

$$e_q^{(k)} = \sum_j \overline{\tilde{\theta}_j^q(\pi(q))} \theta_j^{(k)}. \tag{9}$$

Hence we obtain,

$$\overline{q}^{\otimes k} \otimes e_q^{(k)} = \Phi_{\pi(q)}^{(k)}. \quad \square$$

Let $s_0 : M \to \ell^\times$. In [28], Rawnsley defines a function

$$\eta(x) := \langle e_{s_0(x)}^{(1)} | e_{s_0(x)}^{(1)} \rangle |s_0(x)|^2.$$

It is easy to check that this function is independent of s_0. This function was also studied for Kähler M in [9,10] and in the almost Kähler case in [7]. A short calculation using the previous theorem yields:

Corollary 14. $\eta = \varepsilon^{(1)}$.

3.4. Transition amplitudes

In this section we define the 2-point transition amplitude for symplectic coherent states and show that it can be interpreted as a probability density on M.

Definition 15. The 2-point function, or transition amplitude, is

$$\psi^{(k)}(x, y) := |\langle x^{(k)} | y^{(k)} \rangle|^2 \in C^\infty(M \times M).$$

In terms of the reproducing kernel, the 2-point function is

$$\psi^{(k)}(x, y) = \frac{K^{(k)}(y, x) \cdot K^{(k)}(x, y)}{\varepsilon^{(k)}(x)\varepsilon^{(k)}(y)}.$$

As expected, $\psi^{(k)}(x, x) = 1$. The Cauchy–Schwartz inequality

$$|\langle \Phi_x^{(k)} | \Phi_y^{(k)} \rangle|^2 \leq \| \Phi_x^{(k)} \|^2 \| \Phi_y^{(k)} \|^2$$

implies $\psi^{(k)}(x, y) \in [0, 1]$. Since the map $x \mapsto \mathbb{C} \cdot \Phi_x^{(k)}$ is an embedding for k sufficiently large [24], $x \neq y$ implies $\Phi_x^{(k)} \neq \Phi_y^{(k)}$ so that $\psi^{(k)}(x, y) = 1$ if and only if $x = y$.

Theorem 16. *For each $x \in M_k$, $\psi^{(k)}(x, y)$ is a probability density on M with respect to the coherent measure $\mu^{(k)}(y)$.*

Proof. For each x with $\varepsilon^{(k)}(x) \neq 0$,

$$\int_M \psi^{(k)}(x, y)\, \mathrm{d}\mu^{(k)}(y) = \frac{1}{\varepsilon^{(k)}(x)} \int_M K^{(k)}(y, x) \cdot K^{(k)}(x, y) \epsilon_\omega(y) = 1. \quad \square \tag{10}$$

In [10], Cahen–Gutt–Rawnsley define a 2-point function for Kähler M in terms of Rawnsley-type coherent states:

$$\psi'(x, y) = \frac{|\langle e_q^{(1)} | e_{q'}^{(1)} \rangle|^2}{\|e_q^{(1)}\|^2 \|e_{q'}^{(1)}\|^2}$$

where $x = \pi(q)$ and $y = \pi(q')$. By Theorem 13 we see that $\psi' = \psi^{(1)}$. Moreover, if the quantization is regular (i.e. $\varepsilon^{(k)}(x) = \text{const}$ for all k) then it follows from [10, Proposition 2 and equation 1.7] that $\psi^{(k)} = (\psi^{(1)})^k$.

The transition amplitude can be expressed in terms of peak sections by a simple calculation using Eq. (7):

Theorem 17. $\psi^{(k)}(x, y)\mu^{(k)}(y) = |S_x^{(k)}(y)|^2$

Theorem 16 is therefore equivalent to (8).

Finally, we note here that the association to each $\hbar = 1/k,\ k \in \mathbb{Z}_+$ of $\mathcal{H}_k$, $\mathbb{C} \cdot \Phi_x^{(k)}$ and $\mu^{(k)}$ defines a pure state quantization of the integral symplectic manifold (M, ω) (see [20, p. 113] for the definitions).

4. Berezin–Toeplitz quantization

In this section we study the Berezin quantization [3] induced by the coherent state map $\Phi^{(k)}$. This method of quantization is studied in detail in the context of analytic function spaces in [20]. The extension to Hilbert spaces of sections of the prequantum bundle can be described in terms of symplectic coherent states.

4.1. Overcompleteness and characteristic sets

In this section we consider the most important property of coherent states: overcompleteness.

Definition 18. A system of coherent states $\{|x\rangle \in \mathcal{H}_k \mid x \in M\}$ is overcomplete with respect to a measure μ if

(1) $\langle x|y\rangle \neq 0$ for all $x, y \in M$ with $|x\rangle, |y\rangle \neq 0$, and
(2) $\int_M |x\rangle\langle x| \, \mathrm{d}\mu(x) = \mathbf{1}_{\mathcal{H}_k}$.

Theorem 19. *The system of symplectic coherent states $\{|x^{(k)}\rangle \mid x \in M\}$ defined in Section 3 is overcomplete with respect to the coherent measure $\mu^{(k)}$. In particular,*

$$\int_M |x^{(k)}\rangle\langle x^{(k)}| \, \mathrm{d}\mu^{(k)}(x) = \mathbf{1}_{\mathcal{H}_k}. \tag{11}$$

Proof. We compute, for every $s_1, s_2 \in \mathcal{H}_k$,

$$\begin{aligned}\langle s_1| \int_M |x^{(k)}\rangle\langle x^{(k)}| \, \mathrm{d}\mu^{(k)}(x)|s_2\rangle &= \int_M \langle s_1|\Phi_x^{(k)}\rangle\langle \Phi_x^{(k)}|s_2\rangle \, \epsilon_\omega(x) \\ &= \int_M h(s_1(x), s_2(x)) \, \epsilon_\omega(x) = \langle s_1|s_2\rangle.\end{aligned}$$

so that $\int_M |x^{(k)}\rangle\langle x^{(k)}| \, \mathrm{d}\mu^{(k)} = \mathbf{1}_{\mathcal{H}_k}$ as desired. □

Corollary 20. *There exist points $x_1, \ldots, x_{d_k} \in M$ such that $\{|x_1^{(k)}\rangle, \ldots, |x_{d_k}^{(k)}\rangle\}$ is a basis for $\mathcal{H}_k$.*

Proof. Let $x_1 \in M_k$ and set $\mathcal{S}_1 = \{|x_1^{(k)}\rangle\}$. If $d_k = 1$ then $\mathcal{S}$ is a basis for $\mathcal{H}_k$. Suppose $d_k > r \geq 1$ and let $\mathcal{S}_r = \{|x_1^{(k)}\rangle, \ldots, |x_r^{(k)}\rangle\}$ be a set of linearly independent vectors in $\mathcal{H}_k$. Since $d_k > r$, there is some vector $|\psi\rangle \notin \mathrm{span}_{\mathbb{C}}\mathcal{S}_r$. Suppose for every $x \in M$ that $|x^{(k)}\rangle \in \mathrm{span}_{\mathbb{C}}\mathcal{S}_r$. Then

$$\int_M |x^{(k)}\rangle\langle x^{(k)}|\psi\rangle \mathrm{d}\mu^{(k)}(x) \in \mathrm{span}_{\mathbb{C}} S_r.$$

which implies

$$\int_M |x^{(k)}\rangle\langle x^{(k)}|\psi\rangle \mathrm{d}\mu^{(k)}(x) \neq |\psi\rangle.$$

This contradicts Theorem 19. Hence, there is some $x \in M$ such that $|x^{(k)}\rangle \notin \mathrm{span}_{\mathbb{C}}\mathcal{S}_r$. Let $x_{r+1} = x$. Then $\mathcal{S}_{r+1}$ is a linearly independent set in $\mathcal{H}_k$. We continue inductively. Since $d_k < \infty$, the process must stop, and the resulting set is the required linearly independent set. □

This corollary motivates the following definition, which is a generalization of the characteristic point sets introduced in [2].

Definition 21. A set $\mathcal{S} \subseteq M$ is characteristic if for every $s \in \mathcal{H}_k$,

$$s|_{\mathcal{S}} = 0 \quad \text{implies } s = 0.$$

Theorem 22. *If $\mathcal{S} \subseteq M$ is characteristic, then $\{|x^{(k)}\rangle \mid x \in \mathcal{S}\}$ is complete.*

Proof. If $s(x) = 0$ for all $x \in \mathcal{S}$ implies $s = 0$, then $\langle x^{(k)}|s\rangle = 0$ for all $x \in \mathcal{S}$ implies $s = 0$. Hence, the only vector orthogonal to $\{|x^{(k)}\rangle \mid x \in \mathcal{S}\}$ is 0, which means $\mathcal{S}$ is complete. □

4.2. Quantization

In this section we study the quantizing map $Q : C^\infty(M) \to \mathrm{Op}(\mathcal{H}_k)$ resulting from the overcompleteness relation (11). Applying Berezin's method of quantization [3], we have

Definition 23. The Berezin quantization $Q^{(k)}(f)$ of $f \in C^\infty(M)$ is the operator

$$Q^{(k)}(f) := \int_M f(x)|x^{(k)}\rangle\langle x^{(k)}|\,\mathrm{d}\mu^{(k)}(x).$$

In fact, $Q^{(k)}(f)$ converges for $f \in L^\infty(M)$. Some basic theorems about Berezin's method of quantization of analytic function spaces apply in this case; see for example [20, Theorem 1.3.5]. There is another way to describe the Berezin quantization of f. For each $s \in L^2(M, L^k)$ we have

$$(\Pi_k s)(x) = \int_M \overline{K^{(k)}(x, y)} \cdot s(y)\epsilon_\omega(y) = \langle \Phi_x^{(k)}|s\rangle.$$

Therefore

$$\begin{aligned}(Q^{(k)}(f)s)(x) &= \left(\int_M f(y)|y^{(k)}\rangle\langle y^{(k)}|s\rangle\,\mathrm{d}\mu^{(k)}(y)\right)(x)\\ &= \int_M \overline{K^{(k)}(x, y)} \cdot (f(y)(\Pi_k s)(y))\epsilon_\omega(y) = \left(\Pi_k \circ M_f \circ \Pi_k s\right)(x)\end{aligned}$$

where M_f denotes the multiplication operator. In this form, $Q^{(k)}(f)$ is known as the Toeplitz quantization of f. The Berezin–Toeplitz quantization and Kostant–Souriau quantizations of f are related by Tuynman's formula [30]:

$$\Pi_k \circ Q_{\mathrm{KS}}^{(k)}(f) \circ \Pi_k = Q^{(k)}\left(f - \frac{1}{2k}\Delta f\right)$$

where Δ is the Laplacian associated to the metric $g = \omega(\cdot, J\cdot)$. See [8] for the theory of generalized Toeplitz operators, and [5] for an analysis of the semiclassical properties of $Q^{(k)}$.

We can recast Berezin's covariant symbol [3] in terms of the symplectic coherent states.

Definition 24. The covariant symbol $\widehat{A} \in C^\infty(M)$ associated to the operator $A \in \mathrm{Op}(\mathcal{H}_k)$ is

$$\widehat{A}(x) := \langle x^{(k)}|A|x^{(k)}\rangle,$$

where $A|x^{(k)}\rangle := \sum_{j=1}^{d_k} \overline{\theta_j^{(k)}(x)} \otimes A\theta_j^{(k)}$.

A consequence of Theorem 13 is that Definition 24 agrees with the covariant symbol defined in [9] for Kähler M using Rawnsley-type coherent states. A standard result involving Berezin's covariant symbol is true in our case as well (see [10] for an analogous computation with Rawnsley-type coherent states):

Theorem 25. $\mathrm{Tr}\, A = \int_M \widehat{A}(x)\,\mathrm{d}\mu^{(k)}(x)$.

We conclude this section by pointing out the relationship, in the Spin^c and Kähler cases, between symplectic coherent states and the heat kernel of the appropriate Dirac-type operator. See [4, Chapter 3] for a detailed analysis of the heat kernel, some properties of which we will use below.

The heat kernel $K_t^{(k)} \in C^\infty(\mathbb{R}_+ \times M \times M, \overline{L^k} \boxtimes L^k)$ of the Laplacian associated to $\overline{\partial}_k$ (or $\overline{\partial\!\!\!/}_k$ in the Spin^c case) admits an expansion

$$K_t^{(k)}(x, y) = \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda_j t}\overline{\theta_j^{(k)}(x)} \otimes \theta_j^{(k)}(y) \tag{12}$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ are the eigenvalues of the Laplacian with corresponding eigenmodes $\theta_j^{(k)} \in \Gamma(L^k)$. Moreover,

$$\left| \sum_{j=d_k+1}^{\infty} \mathrm{e}^{-t\lambda_j} \overline{\theta_j^{(k)}} \otimes \theta_j^{(k)} \right| \leq C \mathrm{e}^{-\lambda_{d_k+1} t}$$

for some constant $C > 0$ [4, Proposition 2.37]. Hence, the large time limit of the heat kernel is a symplectic coherent state:

$$\lim_{t\to\infty} K_t^{(k)}(x, y) = \Phi_x^{(k)}(y).$$

In the almost Kähler case, although the heat kernel has an expansion of the form (12), the low lying eigenvalues of the polarized states are not necessarily zero, and so the large time limit of the heat kernel is not directly related to the symplectic coherent states.

Finally, observe that $\mu^{(k)} = \lim_{t\to\infty} \operatorname{Tr} K_t^{(k)}$ and so Theorem 25, applied to the identity operator, yields the familiar index formula $d_k = \int_M \lim_{t\to\infty} \operatorname{Tr} K_t^{(k)}$.

5. Classical and semiclassical behavior

5.1. Classical behavior for finite k

In this section we show that the coherent states defined in Section 3 are the quantum states that behave most classically: they are maximally peaked and evolve classically.

Consider for a moment the almost Kähler quantization of M so that the prequantum bundle L^k is the line bundle $\ell^{\otimes k}$. In this case, the coherent state $\Phi_x^{(k)}$ is the projection of the Dirac distribution onto $\mathcal{H}_k$. To see why, let $x \in M$ and trivialize $\ell^{\otimes k}$ over an open set U containing x with a unit section s_0. Let $\tilde{\delta}_x^{(k)}$ denote the Dirac distribution on U centered at x and define

$$\delta_x^{(k)}(y) = \begin{cases} \tilde{\delta}_x^{(k)}(y)\overline{s_0(x)} \otimes s_0(y) & \text{for } y \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\langle \delta_x^{(k)} | s \rangle = s(x) \quad \forall s \in C^1(M, \ell^{\otimes k}).$$

$\delta_x^{(k)}$ does not depend on our choice of s_0. We now see that $\Phi_x^{(k)}$ is the projection onto $\mathcal{H}_k$ of $\delta_x^{(k)}$ since

$$\left(\Pi_k \delta_x^{(k)}\right)(y) = \int_M \overline{K^{(k)}(y, z)} \cdot \delta_x^{(k)}(z) \epsilon_\omega(z) = \Phi_x^{(k)}(y).$$

We next observe that symplectic coherent states are maximally peaked quantum states. The following result holds for coherent states arising from Kähler, almost Kähler and Spin^c quantization.

Theorem 26. *$\Phi_x^{(k)}$ maximizes $|s(x)|^2$ over all $s \in \overline{\ell_x^{\otimes k}} \otimes \mathcal{H}_k$ with $\|s\|^2 = \varepsilon^{(k)}(x)$.*

Proof. As in Theorem 10 we write

$$|s(x)|^2 = |\langle \Phi_x^{(k)} | s \rangle|^2.$$

This is minimized when we have equality in the Cauchy–Schwartz inequality, which occurs when s is proportional to $\Phi_x^{(k)}$. □

In the almost Kähler case, we can say more: $\Phi_x^{(k)}$ evolves classically. Suppose $f \in C^\infty(M)$ is such that the Hamiltonian vector field X_f is complete. The flow of X_f induces a Hamiltonian evolution of sections in the quantum Hilbert space as follows [31, Section 8.4]:

We can lift the Hamiltonian vector field X_f to a vector field V_f on $T(\ell^{\otimes k})$; that is, there exists a unique vector field on $\ell^{\otimes k}$ defined by $\pi_* V_f = X_f$ and $\frac{1}{k} V_f \lrcorner \alpha = \frac{1}{k} V_f \lrcorner \overline{\alpha} = f \circ \pi$, where α is the connection 1-form on the complement of the zero section of $\ell^{\otimes k}$. If the fiber coordinate is $z = re^{i\phi}$ in a local trivialization, then

$$V_f = X_f + kL \frac{\partial}{\partial \phi}$$

where $L = X_f \lrcorner \tau + f$ is the Lagrangian associated to f by a local symplectic potential τ. Locally, $T\ell^{\otimes k} \simeq TM \times \mathbb{C}$ and we have identified $X_f \in TM$ and $\frac{\partial}{\partial \phi} \in T\mathbb{C}$ with the corresponding vector fields in $TM \times \mathbb{C}$. The flow ξ_t of V_f is fiber preserving and projects to the flow ρ_t of X_f. Moreover, ξ_t induces a linear pull-back action $\widehat{\rho}_t : \Gamma(\ell^{\otimes k}) \to \Gamma(\ell^{\otimes k})$ by

$$\xi_t(\widehat{\rho}_t s(x)) = s(\rho_t x). \tag{13}$$

In fact, this action is infinitesimally generated by the Kostant–Souriau quantization of f:

$$\frac{\mathrm{d}}{\mathrm{d}t} \widehat{\rho}_t = ik Q^{(k)}_{\mathrm{KS}}(f) \widehat{\rho}_t.$$

This is one of the motivations for the Kostant–Souriau quantization $Q^{(k)}_{\mathrm{KS}}$ of f. The restriction of $\widehat{\rho}_t$ to $\mathcal{H}^0_k$ is a 1-parameter unitary group.

Extending the action of ξ_t to $\overline{\ell^{\otimes k}} \boxtimes \ell^{\otimes k}$ in the obvious way, we have:

Theorem 27. *For $f \in C^\infty(M)$ such that X_f is complete,*

$$\xi_t \Phi^{(k)}_x(y) = \Phi^{(k)}_{\rho_t x}(\rho_t y); \tag{14}$$

i.e. the symplectic coherent states evolve classically. Equivalently,

$$\widehat{\rho}_t \Phi^{(k)}_x = \Phi^{(k)}_x.$$

Proof. Since $\widehat{\rho}_t$ is a unitary endomorphism of $\mathcal{H}^0_k$, we have

$$\begin{aligned} \widehat{\rho}_t \Phi^{(k)}_x(y) &= \sum_{j=1}^{d_k} \overline{\widehat{\rho}_t \theta^{(k)}_j(x)} \otimes \widehat{\rho}_t \theta^{(k)}_j(y) \\ &= \sum_{j=1}^{d_k} \overline{\left(\theta^{(k)}_j(x)\right)^t \widehat{\rho}_t} \otimes \widehat{\rho}_t \theta^{(k)}_j(y) \\ &= \Phi^{(k)}_x(y). \quad \square \end{aligned}$$

5.2. The semiclassical limit

The asymptotic analysis of generalized Bergman kernels by Ma–Marinescu reveals the semiclassical behavior of peak sections [24, equation 3.24]. Let $\{r_k\}$ be a sequence of real numbers with $r_k \to 0$ and $\sqrt{k}\, r_k \to \infty$ as $k \to \infty$. Denote by $B(x, r)$ the open geodesic ball of radius r centered at $x \in M$. Then

$$\int_{B(x,r_k)} |S^{(k)}_x(y)|^2 \epsilon_\omega(y) = 1 - O(1/k), \qquad k \to \infty. \tag{15}$$

Comparing this with (8) we see that the peak section $S^{(k)}_x$ is asymptotically concentrated about x. In terms of the transition amplitude, (15) is

$$\int_{B(x,r_k)} \psi^{(k)}(x, y)\, \mu^{(k)}(y) = 1 - O(1/k), \qquad k \to \infty. \tag{16}$$

Combining this with Theorem 16, we have:

Theorem 28. *If* $f \in C^1(M)$ *then*

$$\lim_{k\to\infty} \int_M f(y)\psi^{(k)}(x,y)\mu^{(k)}(y) = f(x);$$

that is, $\lim_{k\to\infty} \psi^{(k)}(x,y)\mu^{(k)}(y) = \delta_x(y)$.

Proof. Let $\{r_k\}$ be a sequence of positive real numbers with $r_k \to 0$ and $\sqrt{k}\, r_k \to \infty$. By Theorem 17 we have, for each $x \in M_k$,

$$\left|\int_M (f(y) - f(x))\psi^{(k)}(x,y)\,\mathrm{d}\mu^{(k)}(y)\right| \le \int_{B(x,r_k)} |f(y) - f(x)|\,|S_x^{(k)}(y)|^2 \epsilon_\omega(y) + \int_{M\setminus B(x,r_k)} |f(y) - f(x)|\,|S_x^{(k)}(y)|^2 \epsilon_\omega(y). \tag{17}$$

The first integral on the right hand side of (17) goes to zero as $k \to \infty$ because of (15) and the fact that f is continuous. The second integral on the right hand side of (17) goes to zero since $f(x) - f(y)$ is bounded (specifically as a function of y) and Eqs. (15) and (8) imply that the peak sections go to zero outside the ball $B(x, r_k)$. □

Of course, there is a physical reason to expect this behavior. The coherent state localized at x should be the quantum state most concentrated about x. In the semiclassical limit, we expect to recover the classical picture — in particular the classical state most concentrated about x is the Dirac distribution at x.

6. Examples

6.1. The complex plane

This example is well known [17,31]. We will take $z = \frac{1}{\sqrt{2}}(x + iy)$, use the standard symplectic form $\omega = i\mathrm{d}z \wedge \mathrm{d}\bar{z}$, and trivialize the prequantum line bundle (globally since $\mathbb{C}$ is contractible) with the symplectic potential $\tau = \frac{i}{2}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z)$. The space $\mathcal{H}_k = \{f(z)\mathrm{e}^{-k|z|^2/2}\}$ of polarized sections of $\ell^{\otimes k}$ relative to the standard complex structure can be identified with a weighted Bargmann space [2]. A unitary basis for $\mathcal{H}_k$ is $\{\sqrt{\frac{k}{j!}}\, z^j \mathrm{e}^{-k|z|^2/2}\}_{j\in\mathbb{N}}$. The reproducing kernel is the usual Bergman kernel

$$K^{(k)}(w,z) = k\sum_{j=0}^{\infty} \frac{(\bar{w}z)^j}{j!}\mathrm{e}^{-k|z|^2/2 - k|w|^2/2} = k\mathrm{e}^{\bar{w}z - k|z|^2/2 - k|w|^2/2}.$$

The coherent density is $\varepsilon^{(k)} = k$ and the 2-point function is $\mathrm{e}^{-k|z-w|^2}$. In this case, it is easy to see that the semiclassical limit yields the expected results:

$$\lim_{k\to\infty} \psi^{(k)}(w,z)\mu^{(k)}(z) = \lim_{k\to\infty} \left(\frac{k}{2\pi}\right)\mathrm{e}^{-k|w-z|^2}\omega = \delta(w - z)\omega.$$

6.2. The 2-sphere

Coherent states on S^2 are constructed by Perelomov in [27] using Lie group techniques. As we will see, the construction of Section 3 yields the same results without using any group structure. The correspondence between the two methods is due to Theorem 4 and the fact that Perelomov's coherent states are reproducing.

We trivialize $S^2 \simeq \mathbb{C}P^1 = \{[z_0, z_1]\}/\mathbb{C}$ over the open set $U_0 = \{[z_0, z_1] \mid z_0 \neq 0\}$. Define a local coordinate $z = z_1/z_0$. The Fubini-Study symplectic form on U_0 is

$$\omega = \frac{i\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}.$$

Trivializing $\ell^{\otimes k}$ with the symplectic potential $\tau = (1+|z|^2)^{-1} i\bar{z}\mathrm{d}z$, the Hermitian form on $\ell^{\otimes k}$ is given by $h(p,q) = (1+|z|^2)^{-k}\overline{p}q$. We then have the following unitary basis for $\mathcal{H}_k$:

$$\left\{ \sqrt{(k+1)\binom{k}{j}}\, z^j \mid 0 \le j \le k \right\}.$$

The coherent state localized at w is therefore

$$\Phi_w^{(k)}(z) = (k+1)\sum_{j=0}^{k} \binom{k}{j} (\bar{w}z)^j = (k+1)(1+\bar{w}z)^k.$$

The corresponding coherent density is $\varepsilon^{(k)} = k+1$; one must take care to include the extra factors arising from the Hermitian structure when evaluating $\Phi_w^{(k)}(w)$. In this case, Theorem 28 becomes

$$\lim_{k\to\infty} \psi^{(k)}(w,z)\mu^{(k)}(z) = \lim_{k\to\infty} \frac{k+1}{2\pi}\left[\frac{(1+\bar{w}z)(1+w\bar{z})}{(1+|z|^2)(1+|w|^2)}\right]^k \frac{i\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1+|z|^2)^2} = \delta(w-z).$$

6.3. The 2-torus

For $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$, let $T^2(\lambda) = \mathbb{C}/\{m+n\lambda \mid m,n\in\mathbb{Z}\}$. λ is known as the modulus of the torus. The standard symplectic form $\omega = 2\pi i\lambda_2^{-1}\mathrm{d}z\wedge\mathrm{d}\bar{z}$ on $\mathbb{C}$, normalized to be integral on $T^2(\lambda)$, descends to a symplectic form on $T^2(\lambda)$. The prequantum line bundle $\ell^{\otimes k}$ can be lifted to a line bundle over $\mathbb{C}$ (since $\mathbb{C}$ is the universal cover of $T^2(\lambda)$). The resulting line bundle can be globally trivialized. Hence we will identify sections of $\ell^{\otimes k}$ with appropriately pseudoperiodic functions on $\mathbb{C}$.

If we trivialize with the symplectic potential $\tau = i\pi\lambda_2^{-1}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z)$ then the Hermitian form is given by $h(p,q) = \overline{p}q$. A unitary basis for the quantum Hilbert space can be given in terms of ϑ-functions. Let

$$\vartheta_j(\lambda; z) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{i\lambda\pi(kn^2+2jn)+2\pi i\sqrt{2}(j+kn)z},$$

$$\psi_j^{(k)}(z,\bar{z}) = \mathrm{e}^{k\pi z(z-\bar{z})/\lambda_2}\vartheta_j(\lambda; z), \quad \text{and}$$

$$N_{k,j} = \|\psi_j^{(k)}\|^2 = \frac{1}{\sqrt{2k\lambda_2}}\mathrm{e}^{2\pi j^2\lambda_2/k}.$$

A unitary basis for $\mathcal{H}_k$ is $\{N_{k,j}^{-1/2}\psi_j^{(k)}(z,\bar{z})\}_{j=0}^{k-1}$. From this we construct the coherent state localized at $w = \frac{1}{\sqrt{2}}(w_1 + iw_2)$:

$$\Phi_w^{(k)}(z) = \sum_{j=0}^{k-1}\sqrt{2k\lambda_2}\,\mathrm{e}^{-2\pi j^2\lambda_2/k}\mathrm{e}^{\sqrt{2}\pi ik(zy-\bar{w}w_2)/\lambda_2}\overline{\vartheta_j(\lambda; w)}\vartheta_j(\lambda; z).$$

The coherent density is

$$\varepsilon^{(k)}(z) = \sqrt{2k\lambda_2}\mathrm{e}^{-2\pi ky^2/\lambda_2}\sum_{j=0}^{k-1}\mathrm{e}^{-2\pi j^2\lambda_2/k}|\vartheta_j(\lambda; z)|^2.$$

The semiclassical limit (Theorem 28) yields the identity

$$\begin{aligned}\delta(w-z) &= \lim_{k\to\infty}\psi^{(k)}(w,z)\mu^{(k)}(z)\\ &= \lim_{k\to\infty}\sqrt{2k\lambda_2}\,\mathrm{e}^{-2\pi kw_2^2/\lambda_2}\left(\sum_{j=0}^{k-1}\mathrm{e}^{-2\pi j^2\lambda_2/k}|\vartheta_j(\lambda; z)|^2\right)^{-1}\\ &\quad\cdot\sum_{j,l=0}^{k-1}\mathrm{e}^{-2\pi(j^2+l^2)\frac{\lambda_2}{k}}\overline{\vartheta_j(\lambda; w)\vartheta_l(\lambda, z)}\vartheta_j(\lambda; z)\vartheta_l(\lambda; w)\,\frac{2\pi i}{\lambda_2}\mathrm{d}z\wedge\mathrm{d}\bar{z}.\end{aligned}$$

6.4. Higher genus Riemann surfaces

We will construct coherent states on a compact Riemann surface Σ_g of genus $g \geq 2$ by uniformizing Σ_g as the quotient of the complex upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by a Fuschian group Γ (see [1,16] for details). The coherent states in this section correspond to those of Klimek–Lesniewski [19, equation 4.5].

Let $\Gamma < PSL(2, \mathbb{Z})$ be a Fuschian group. $PSL(2, \mathbb{Z})$, and hence Γ, acts on $\mathfrak{H}$ by fractional linear transformations. The space $\Sigma_g := \Gamma \backslash \mathfrak{H}$ is a compact manifold if and only if Γ is a hyperbolic group, which we will henceforth assume. The Kähler form $\omega = i(\operatorname{Im} z)^{-2} \mathrm{d}z \wedge \mathrm{d}\bar{z}$ descends to a symplectic form on Σ_g, as do the complex structure and Kähler metric.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathfrak{H}$ define $j(\gamma, z) := cz + d$. An automorphic form of weight k relative to Γ is a function $f : \mathfrak{H} \to \mathbb{C}$ such that $f(\gamma z) = j(\gamma, z)^k f(z)$. The space $\mathcal{A}_k(\mathfrak{H})$ of automorphic forms on $\mathfrak{H}$ of weight k relative to Γ has an Hermitian product

$$\langle f|g\rangle_k = \int_{\mathfrak{H}} \overline{f(z)} g(z) (\operatorname{Im} z)^k \frac{\omega}{2\pi}.$$

The bundle $\ell := T^* \Sigma_g^{(1,0)}$ is a prequantum bundle for Σ_g since its curvature is $-i\omega$. The quantum Hilbert space $\Gamma(\Sigma_g, \mathcal{O}(T^* \Sigma_g^{(1,0)}))$ is isomorphic to $\mathcal{A}_2(\Sigma_g)$. Sections of $\ell^{\otimes k}$ correspond to automorphic forms of weight $2k$ relative to Γ restricted to Σ_g. The Hermitian form descends to Σ_g and is known as the Weil–Petersson inner product.

The reproducing kernel for $\mathcal{A}_{2k}(\mathfrak{H})$ is known (see for example [26]) and descends to Σ_g via a Poincaré series. The resulting coherent state localized at w is

$$\Phi_w^{(k)}(z) = \frac{k-1}{2} \sum_{\gamma \in \Gamma} \left(\frac{2i}{\gamma z - \bar{w}} \right)^k j(\gamma, z)^{-k}.$$

The associated coherent density is

$$\varepsilon^{(k)}(z) = (k-1) \sum_{\{\gamma, \gamma^{-1}\} \subset \Gamma} \operatorname{Re} \left[\frac{2i}{(\gamma z - \bar{z}) j(\gamma, z)} \right]^{2k}.$$

Finally, the semiclassical limit of Theorem 28:

$$\begin{aligned}
\delta(w - z) &= \lim_{k \to \infty} \psi^{(k)}(w, z) \mu^{(k)}(z) \\
&= \lim_{k \to \infty} \frac{k-1}{2\pi} 4^{k-1} (\operatorname{Im} z)^{k-2} (\operatorname{Im} w)^k \left(\sum_{\{\gamma, \gamma^{-1}\} \subset \Gamma} \operatorname{Re} \left[\frac{2i}{(\gamma z - \bar{z}) j(\gamma, z)} \right]^{2k} \right)^{-1} \\
&\quad \cdot \sum_{\gamma, \gamma' \in \Gamma} (|\gamma z - \bar{w}| \, |j(\gamma, z)|)^{-2k} \, i \mathrm{d}z \wedge \mathrm{d}\bar{z}.
\end{aligned}$$

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